Complexe A~~a~~nalyse Midtoets 03-10-08

Blad 1/

$\boxed{1}$ $f(z) = \bar{z}^2 + 2\bar{z}$ $\qquad z \equiv x+iy$

$= (x-iy)^2 + 2(x-iy)$

1 10 $= x^2-y^2-2xyi + 2x - 2iy$

2 9 $= x^2-y^2+2x \quad -i(2xy+2y)$

3 8 $= u(x,y) + i\,v(x,y)$ met $u(x,y) = x^2-y^2+2x$

4 10 $\qquad v(x,y) = -2xy-2y$

5 9 $\frac{\partial u}{\partial x} = 2x+2 \qquad \frac{\partial v}{\partial y} = -2x-2$

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$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial x} = -2y$ ✓

Een functie $f(z) = u(x,y) + iv(x,y)$ is analytisch in een punt als daar geldt:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{en} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = 2x+2 = -2x-2 = \frac{\partial v}{\partial y}$$

$2x+2 = -2x-2 \qquad 4x = -4 \Rightarrow x = -1$

$$\frac{\partial u}{\partial y} = -2y = -(-2y) = -\frac{\partial v}{\partial x}$$

$2y = -2y \Rightarrow y = 0$

Dus $f(z) = \bar{z}^2 + 2\bar{z}$ is alleen analytisch in het punt $z=-1$ $(z=(-1,0))$ ✓

$\boxed{2}$ $i^z \equiv e^{z\log i} = e^{z[\mathrm{Log}|i| + i\mathrm{Arg}(i) + i2k\pi]}$ $\qquad k \in \mathbb{Z}$

$= \exp(z[\mathrm{Log}\, i + i\mathrm{Arg}(i) + i2k\pi])$

$= \exp(zi[\pi/2 + 2k\pi])$

Ik kies de 'principal value' van de log, dus $k=0$:

$i^z = \exp(zi\,\pi/2)$ en dus

$$\frac{d}{dz} i^z = \frac{d}{dz} \cancel{\exp}\, e^{iz\pi/2} = i\pi/2\, e^{i\pi/2 z} = \frac{\pi}{2}\, i^{1+z}$$

(Voor een andere keuze van ~~de~~ 'branch' van de log moet het argument dus t.o.v. een andere τ bepaald worden, met $(\tau, \tau+2\pi]$ het interval waarop de hoeken in de e-macht ~~gede~~ liggen) ✓

$\boxed{3}$ $u(x,y) = x^3 - 3xy^2 + 2y$ $\qquad z \equiv x+iy \qquad x = \text{Re}\, z = \frac{z+\bar{z}}{2}$

$$y = \text{Im}\, z = \frac{z-\bar{z}}{2i}$$

~~$x^3 - 3xy^2 + 2y = \left(\frac{z+\bar{z}}{2}\right)^3 - 3\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right)^2 + 2\,\frac{z-\bar{z}}{2i}$~~

~~$= \left(\frac{z^2+2z\bar{z}+\bar{z}^2}{4}\right)\left(\frac{z+\bar{z}}{2}\right) - 3\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z^2-2z\bar{z}+\bar{z}^2}{-4}\right) + 2\,\frac{z-\bar{z}}{2i}$~~

~~$= \frac{z^3+z^2\bar{z}+2z^2\bar{z}+2z\bar{z}^2+\bar{z}^2z+\bar{z}^3}{8}$~~

~~$+ 3\,\frac{z^3-2z^2\bar{z}+z\bar{z}^2+\bar{z}z^2-2z\bar{z}^2+\bar{z}^3}{4}$~~

~~$+ 2\,\frac{z-\bar{z}}{2i}$~~

~~$= \frac{1}{8}(z^3+3z^2\bar{z}+3z\bar{z}^2+\bar{z}^3) + \frac{3}{4}(z^3-z^2\bar{z}-z\bar{z}^2+\bar{z}^3) + \frac{z-\bar{z}}{i}$~~

$$z^3 = (x+iy)^3 = \sum_{k=0}^{3}\binom{3}{k}x^k(iy)^{3-k} = (iy)^3 + 3x(iy)^2 + 3x^2(iy) + x^3$$
$$= -y^3i + 3x^2iy \quad -3xy^2 + x^3$$

z^3 ~~[illegible]~~ $-2iz$ $= x^3 - 3xy^2$ ~~m~~ $-2i(x+iy) \quad + i(3x^2y - y^3)$

$$= x^3 - 3xy^2 + 2y \quad + i(3x^2y - y^3 - 2x)$$
$$= u(x,y) + iv(x,y)$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \quad \overset{\checkmark}{=} \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2 \qquad \text{voldoet } \forall x,y \in \mathbb{R}$$

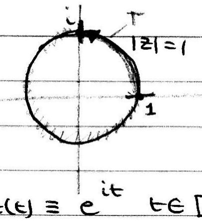
$$\frac{\partial u}{\partial y} = -6xy + 2 \quad \overset{\checkmark}{=} \; -\frac{\partial v}{\partial x} = -(6xy-2) = -6xy+2 \qquad \text{voldoet } \forall x,y \in \mathbb{R}$$

Dus $u(x,y) = \text{Re}[z^3 - 2iz]$ is dus het reële deel van een analytische functie.

Het ~~analy~~ imaginaire deel van deze functie is dus

$v(x,y) = 3x^2y - y^3 - 2x + \text{const}$

$\boxed{4}$



$\Gamma \equiv \{ z \mid z = e^{it} \text{ met } t \in [0, \pi/2] \}$

$$\int_\Gamma e^{1/z}\,dz = \int_\Gamma f(z)\,dz \qquad f(z) = e^{1/z}$$

[$z(t) \equiv e^{it} \quad t \in [0, \pi/2]$, dan $\qquad$ maar er wordt een bovengrens
$\int_\Gamma e^{1/z}\,dz = \int_{t=0}^{\pi/2} e^{e^{-it}}\, ie^{it}\,dt$ $\qquad$ gevraagd]

Blad 2 →

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Blad 2/

$$\left|\int_\Gamma e^{1/z}\,dz\right| = \int_\Gamma |e^{1/z}|\,dz$$

$$e^{1/z} = e^{\frac{1}{x+iy}} = \exp\left(\frac{1}{x+iy}\cdot\frac{x-iy}{x-iy}\right) = \exp\left(\frac{x-iy}{x^2+y^2}\right)$$

$x^2+y^2=1$, omdat Γ ~~de~~ een kromme is waarvoor geldt $|z|^2 = x^2+y^2 = 1^2 = 1$

Dus

$$e^{1/z} = e^{x-iy} = e^{x}e^{-iy}$$

$$|e^{1/z}| = |e^{x}e^{-iy}| = |e^{x}||e^{-iy}| = e^{x}$$

$$\left|\int_\Gamma e^{1/z}\,dz\right| = \int_\Gamma |e^{1/z}|\,dz \leq \max_\Gamma(|e^{1/z}|)\cdot \ell(\Gamma)$$

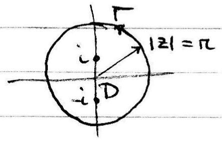
$$= \cancel{e^{x}\cdot\frac{\pi}{2}}\ \max_\Gamma(e^{x})\cdot \ell(\Gamma)$$

$$= e\cdot\frac{\pi}{2}$$

Dus $\left|\int_\Gamma e^{1/z}\,dz\right| \leq e\cdot\frac{\pi}{2}$ ✓

$\boxed{5}$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(z_0)}{(z-z_0)^{n+1}}dz$$



Hier:

$\int_\Gamma \frac{\sin z}{(z+i)^3}\,dz$ $\quad f(z_0) = \sin(z_0)$ $\quad n+1=3,\ n=2$

$z_0 = -i$ $\quad -i \in D$

Dus

$$\int_\Gamma \frac{\sin z}{(z+i)^3}\,dz = \frac{2\pi i}{2!}\ \sin^{(2)}(\overset{z}{\cancel{\ \ }})\Big|_{z=-i} = \pi i\ \cancel{\sin}\ -\sin(z)\Big|_{z=-i} = \pi i\,\sin(i)$$

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